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# Weak and singular solutions of the wave equation in curved spacetime

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**Abstract**
We construct a weak formulation of the wave equation in curved spacetime for solutions which are regularly discontinuous across a hypersurface. We adopt the framework of distributions and tensor-distributions, and allow the presence of discontinuity for the first and second derivatives of the spacetime metric. We thus find the corresponding compatibility conditions to hold at the interface, as replacement for the differential equation, which is undefined there. In particular, we find out that if discontinuity of the first derivatives of the metric is present, such compatibility conditions also involve the mean values of the field, and not only its jump across the discontinuity hypersurface. We also consider the case of singular solutions with support on a hypersurface, and derive the corresponding compatibility conditions. Applications to electromagnetism are presented.

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## 1. Introduction

The wave operator over a pseudo-Riemannian manifold and for a generic tensor was defined by Lichnerowicz in [1], with the name of generalized Laplace operator, generalizing the analogous definition of de Rham, which instead applies only to antisymmetric tensors [2]. The generalized Laplace operator reduces to the usual D'Alembert operator only for ordinary functions and in a flat spacetime.

In this paper, we consider the problem of matching two solutions of the wave equation across a discontinuity hypersurface $\Sigma$ of the spacetime. To study this problem, one has to introduce some compatibility conditions in order to replace the wave equation on the hypersurface, where it is not defined. The set given by the ordinary wave equation, to hold at each side of $\Sigma$, plus compatibility conditions to hold on the hypersurface, defines a weak solution of the wave equation, in a sense which depends on the choice of the compatibility conditions [3].

The theory of distributions and tensor-distributions [1, 4] provides a general method to define, in an axiomatic way, tensor weak solutions of arbitrary differential equations which are regularly discontinuous [5]. In the general case, it suffices in fact to replace the discontinuous fields with the corresponding integrable distributions and to interpret the differential operators contained in the field equations in the sense of distributions. In the case of the wave equation, for example, we find that the corresponding distributional equation is equivalent to the ordinary one on each side of the discontinuity hypersurface, plus some compatibility conditions to hold at the interface, i.e. exactly what we were looking for to define weak solutions of the wave equation. Lichnerowicz's framework permits us to use a very general ambient spacetime, where the eventual presence of discontinuity for the first and second derivatives of the metric (i.e. gravitational waves and shock waves) is allowed [4]. Such generalization is important, since it turns out that the characteristic hypersurfaces of the wave equation are the same as those of the Einstein equations, i.e. lightlike hypersurfaces; thus discontinuity for a solution of the wave equation and for the spacetime metric can coexist. In case discontinuity of the first derivatives of the metric is present, compatibility conditions for the solutions of the wave equation turn out to involve the curvature tensor-distribution, the jump of the field, and also its arithmetic mean (theorem 1).

Weak solutions in the sense of distributions then lead in a natural way to a further generalization, corresponding to solutions of the wave equation with support on a singular hypersurface. Also in this case we can find the corresponding compatibility conditions for the singular component of the field (theorem 2).

The framework of electromagnetism (section 6) permits us to successfully test the method and then to apply it to a series of problems involving discontinuous and singular charge-current distributions.

## 2. Discontinuous fields and distributions in curved spacetime

### *2.1. The curved spacetime*

Let $V_4$ be the spacetime of general relativity, i.e. an oriented differentiable manifold of dimension 4, class ($C^1$, piecewise $C^3$), provided with a strictly hyperbolic metric of signature $--+++$ and class ($C^0$, piecewise $C^2$); let us denote with $\eta_{\alpha\beta\gamma\delta}$ the unit volume 4-form which orients the spacetime and $\nabla$ the associated covariant derivative. Let $\Omega \subset V_4$ be an open connected subset with compact closure. Let units be chosen in order to have the speed of light in empty space $c \equiv 1$. Greek indices run from 0 to 3; Latin indices run from 1 to 3.

The Riemann curvature tensor $R$ is defined by the Ricci formula:

$$(\nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta) V^\sigma = R_{\alpha\beta\rho}{}^{\sigma} V^\rho. \tag{1}$$

The symmetric Ricci tensor is here defined by $R_{\beta\rho} = R_{\sigma\beta\rho}{}^{\sigma}$ and the curvature scalar by $R = R_\alpha{}^\alpha$; the Einstein tensor is $G_{\alpha\beta} = R_{\alpha\beta} - (1/2) R g_{\alpha\beta}$.

For an introduction to Einstein's theory of gravitation in terms of local differential geometry, see, e.g., [6–8].

### *2.2. Regularly discontinuous fields*

Let us recall some general properties of regularly discontinuous functions and tensors (for details see, e.g., [9–12]).

Let $\Sigma \subset \Omega$ be a regular hypersurface of equation $f(x) = 0$; let $f$ have non-vanishing gradient on $\Sigma$. Let $\Omega^+$ and $\Omega^-$ denote the subdomains distinguished by the sign of $f$.

A field $\varphi$ is said to be regularly discontinuous on $\Sigma$ if there exist a pair of regular functions $\Phi^+, \Phi^- \in C^k(\Omega)$ ($k \geqslant 0$) such that their restrictions to the two subdomains $\Omega^+$ and $\Omega^-$, respectively, coincide with those of $\varphi$:

$$\varphi|_{\Omega^+} = \Phi^+|_{\Omega^+}, \qquad \varphi|_{\Omega^-} = \Phi^-|_{\Omega^-}.$$

If $\varphi$ is regularly discontinuous, its restrictions have a finite limit $\varphi^\pm$ for $f \longrightarrow 0^\pm$, the limit being independent of the choice of the path approaching any given $x \in \Sigma$. Then its jump $[\varphi] = \varphi^+ - \varphi^-$ across $\Sigma$ and its arithmetic mean value $\overline{\varphi} = (\varphi^+ + \varphi^-)/2$ are well defined. In the particular case when $\varphi$ is continuous across $\Sigma$, we obviously have $[\varphi] = 0, \overline{\varphi} = \varphi|_\Sigma$.

Let the first and second partial derivatives of the metric be regularly discontinuous on $\Sigma$. Let $f \in C^1(\Omega) \cap C^3(\Omega\backslash\Sigma)$, and let second and third derivatives of $f$ be regularly discontinuous on $\Sigma$. Finally, let $\ell_\alpha \equiv \partial_\alpha f$ denote the gradient of $f$. We have $\ell \neq 0$ for any $x \in \Sigma$.

It is easy to check that the jump of the product of two functions $\varphi$ and $\psi$ obeys the following useful formula:

$$[\varphi\psi] = [\varphi]\overline{\psi} + \overline{\varphi}[\psi]. \tag{2}$$

The jump of a regularly discontinuous function has support on $\Sigma$, but it can be extended off $\Sigma$ for operational reasons, with the help of regular prolongations; by definition a regular prolongation of $[\varphi]$ in $\Omega$ is $\Phi^+ - \Phi^-$, and any function which coincides on $\Sigma$ with $\Phi^+ - \Phi^-$ is also an admissible regular prolongation of $[\varphi]$. Clearly, only the restriction to $\Sigma$ of the prolonged jump is independent of the prolongation.

Similar, if $k > 0$, and if the derivatives of $\varphi$ up to order $k$ are also regularly discontinuous, an admissible regular prolongation must have derivatives which coincide on $\Sigma$ with those of $\Phi^+ - \Phi^-$. Thus it is rather natural to define the partial derivative of the jump as the jump of the partial derivative of the function (see [11, 12]), which in practice coincides with the restriction to $\Sigma$ of the derivative of any prolonged jump. This permits us to handle derivatives of objects of the kind $[\varphi]\delta$, where $\delta$ is a singular distribution, in a compatible way, as we will see. In particular, with this definition the derivative of the jump of a continuous field is not null, unless the field is also $C^1$ (while a derivative tangent to $\Sigma$, which is effected only by the restriction to $\Sigma$ of a field, is instead obviously null when applied to the jump of a continuous field).

Similarly, we can define in an invariant way on $\Sigma$ the partial derivative of the mean value of a function as the mean value of the partial derivative. More generally, here we define the partial derivative of the restriction to $\Sigma$ of a given regular field as the restriction of the partial derivative of the field.

Finally, we define the covariant derivative of the jump of a regularly discontinuous field by means of the mean value $\overline{\Gamma}_{\beta\rho}{}^\sigma$ of the Christoffel symbols which, with our continuity assumptions on the metric, are regularly discontinuous on $\Sigma$. For the jump of a regularly discontinuous vector, for example, with this definition one has that the jump of the covariant derivative is different than the covariant derivative of the jump. By (2) in fact we have

$$\nabla_\alpha[V^\beta] = [\partial_\alpha V^\beta] + \overline{\Gamma}_{\alpha\sigma}{}^\beta[V^\sigma] = [\nabla_\alpha V^\beta] - \left[\Gamma_{\alpha\sigma}{}^\beta\right]\overline{V}^\sigma, \tag{3}$$

and similarly for the jump of a regularly discontinuous tensor.

The covariant derivative of the restriction to $\Sigma$ of a given regular tensor field is defined in the same way, i.e. by means of $\overline{\Gamma}$.

### *2.3. Distributions and tensor-distributions*

Let us recall the basic properties of distributions and tensor-distributions; for complete details see, e.g., [1, 4, 10, 13].

Let $\{x^\alpha\}$, $\alpha = 0, 1, 2, 3$, be a local chart of domain $\Omega \subset V_4$. Let $(T, U)(x)$ denote the scalar product with respect to the metric $g$ of the spacetime of two $p$-tensors $T$ and $U$ at the point $x \in V_4$; in the domain $\Omega$ we have

$$(T, U)(x) = \left(T_{\alpha_1\cdots\alpha_p} U^{\alpha_1\cdots\alpha_p}\right)(x), \quad (x \in \Omega). \tag{4}$$

Let us define integration over $V_4$ by means of the natural volume element $\eta$ of the manifold; locally, we have

$$\eta|_\Omega = \sqrt{|g|}\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^3, \tag{5}$$

and the integral of a summable function $f$ is

$$\int_{V_4} f = \int_{V_4} f(x)\eta(x). \tag{6}$$

Let $D^p(V_4)$ be the space of $p$-tensors $U$ of $V_4$ of class $C^k$, $k \geqslant 2$, and with compact support $S(U)$. If $T$ is a locally integrable $p$-tensor over $V_4$, i.e. if $(T, U)$ is integrable for any $U \in D^p(V_4)$, then we define

$$\langle T, U\rangle = \int_{\bar{V}_4} (T, U). \tag{7}$$

A $p$-tensor-distribution $T$ of $V_4$ is by definition a continuous linear form, with scalar values, over $D^p(V_4)$. Here continuity is intended in the following sense (see, e.g., [1, 4, 13]): given a compact set $K$, consider a sequence $U_i$, $i = 1, 2, \ldots$ of elements of $D^p(V_4)$ with supports $S(U_i) \subset K$, such that $U_i$ and their derivatives of order $\leqslant k$ converge uniformly to 0 for $i \longrightarrow \infty$; $T$ is continuous if $T(U_i) \longrightarrow 0$ for all these sequences, the support belonging to an arbitrary compact $K$.

Locally, in the domain $\Omega$ of a local chart, a generic $p$-tensor-distribution $T$ in $\Omega$ has components $T_{\alpha_1\cdots\alpha_p}$ which are scalar-distributions in $\Omega$; conversely, given, over $\Omega$, $n^p$ scalar-distributions $T_{\alpha_1\cdots\alpha_p}$ they define a $p$-tensor-distribution $T$ in a unique way (see [1, 4]).

The support of a tensor-distribution on $V_4$ is the smallest closed set $S$ in $V_4$ outside which $T$ is identically zero (i.e., it is zero for all test tensors with support outside $S$).

A locally summable $p$-tensor $V$ defines in a natural way an associated distribution $V^D$:

$$V^D(U) = \langle V, U\rangle, \qquad U \in D^p(V_4). \tag{8}$$

Distributions and tensor-distributions which can be constructed this way are called regular, or integrable; those which cannot, are called singular. Since the space of tensor-distributions includes integrable and singular tensor-distributions, it obviously is an extension of the space of ordinary locally integrable tensors (which instead correspond to integrable tensor-distributions only).

Given a scalar-distribution $u$ and a $p$-tensor $V$ of class $C^h$, their product is the $p$-tensor-distribution defined by

$$(uV)(U) = (Vu)(U) = u((V, U)), \qquad U \in D^p(V_4). \tag{9}$$

The covariant derivative of a $p$-tensor-distribution $T$ is the $(p + 1)$-tensor-distribution defined by

$$(\nabla T)(U) = -T(\mathrm{Div}\, U), \tag{10}$$

where $(\mathrm{Div}\, U)^{\alpha_1\cdots\alpha_p} = \nabla_\beta U^{\beta\alpha_1\cdots\alpha_p}$. With the definition above, the classical properties of the covariant derivative hold also for tensor-distributions. Moreover, with definitions (9) and (10) the usual chain rule for the derivation of a product holds.

### 2.4. Distributions associated with a hypersurface

Let us restrict ourselves to a contractile domain $\Omega \subset V_4$, with compact closure, divided by $\Sigma$ into two subdomains $\Omega_\Sigma^+$ and $\Omega_\Sigma^-$, corresponding respectively to $f > 0$ and $f < 0$. Let $\chi_\Sigma^+$ (respectively $\chi_\Sigma^-$) be the function over $\Omega$ that is equal to 1 (respectively 0) over $\Omega_\Sigma^+$ and to 0 (respectively 1) over $\Omega_\Sigma^-$. These functions define in a natural way over $\Omega$ a pair of scalar-distributions, again denoted as $\chi_\Sigma^+$ and $\chi_\Sigma^-$, by means of the following formulae:

$$\chi_\Sigma^\pm(\varphi) = \int_\Omega \chi_\Sigma^\pm \varphi = \int_{\Omega_\Sigma^\pm} \varphi, \qquad \varphi \in D^0(\Omega). \tag{11}$$

One can introduce the so-called class of Leray of $n$-forms associated with $f$ [1, 4], i.e. the $n$-forms $\omega$ such that

$$\eta = \ell \wedge \omega. \tag{12}$$

It can be shown [1, 4] that for $\varphi \in D^0(\Omega)$ the integral

$$\int_{\partial\Omega_\Sigma^-} \varphi\omega = -\int_{\partial\Omega_\Sigma^+} \varphi\omega \tag{13}$$

has a value independent of the choice of $\omega$ satisfying (12). This leads to the following definition of hypersurface integral,

$$\int_\Sigma \varphi = \pm \int_{\partial\Omega_\Sigma^\mp} \varphi\omega, \tag{14}$$

which is independent of the signature or the eventually degenerate kind of $\Sigma$. Note that in a chart adapted to $\Sigma$, i.e. with coordinates $x^\alpha$ such that $x^0 = f$ and thus $\ell_\alpha = {\delta_0}^\alpha$, one has that the form

$$\omega = \sqrt{|g|}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \tag{15}$$

is associated with $f$ in the sense of Leray, as one would expect.

The Dirac measure distribution associated with $\Sigma$ is the distribution $\delta_\Sigma$ defined by

$$\delta_\Sigma(\varphi) = \int_\Sigma \varphi, \qquad \varphi \in D^0(\Omega) \tag{16}$$

where $\delta_\Sigma$ is a singular distribution and has support over $\Sigma$.

As for the derivative of $\delta_\Sigma$, it is possible to prove [1, 4] that there is another distribution with support on $\Sigma$, usually denoted as $\delta'_\Sigma$, such that

$$\nabla\delta_\Sigma = \ell\delta'_\Sigma. \tag{17}$$

In adapted coordinates one simply has $\delta'_\Sigma = \nabla_0\delta_\Sigma$.

A very useful property, which as yet seems to have been neglected in the literature, is that $\delta_\Sigma$ and $\delta'_\Sigma$ are independent distributions in the sense that if a distribution is defined by means of a linear combination of $\delta_\Sigma$ and $\delta'_\Sigma$, such a combination is unique. To see this, let us consider adapted coordinates; if $\gamma$ is a regular function of class $C^h(\Omega)$, $h \geqslant 2$, we have

$$\gamma\delta_\Sigma = 0 \quad \Longleftrightarrow \quad \gamma|_\Sigma = 0. \tag{18}$$

In fact $\gamma\delta_\Sigma = 0$ means $\int_\Sigma \gamma\varphi$ for any test function $\varphi \in D^0(\Omega)$. We similarly have

$$\gamma\delta'_\Sigma = 0 \quad \Longleftrightarrow \quad \gamma|_\Sigma = (\nabla_0\gamma)|_\Sigma = 0. \tag{19}$$

In fact $\gamma\delta'_\Sigma(\varphi) = -\delta_\Sigma(\nabla_0\gamma\varphi + \gamma\nabla_0\varphi)$, and again, by the arbitrariness of $\varphi$ we obtain (19) (it suffices to consider the class of test functions $\varphi$ such that $\varphi|_\Sigma = 0$ and then those such that $\nabla_0\varphi|_\Sigma = 0$). Clearly, the hypothesis $\gamma \in C^h(\Omega)$ can be replaced with $\gamma$ to be of class $C^h$

in some neighbourhood of $\Sigma$, so that the values on $\Sigma$ of $\gamma$ are well defined. Now consider a distribution $F$ given by a linear combination of the following kind,

$$F = \gamma\delta_\Sigma + \beta\delta'_\Sigma, \tag{20}$$

where $\gamma, \beta \in C^h(\Omega)$. Then, we have

$$F = 0 \quad \Longleftrightarrow \quad \gamma|_\Sigma = \beta|_\Sigma = 0. \tag{21}$$

In fact by definition we have

$$F(\varphi) = \int_\Sigma ((\gamma - \nabla_0\beta)\varphi - \beta\nabla_0\varphi). \tag{22}$$

Thus again by arbitrariness of $\varphi$ if $F = 0$ we necessarily have $\beta|_\Sigma = 0$. We consequently have $F = \gamma\delta_\Sigma = 0$ and thus $\gamma|_\Sigma = 0$. In particular, it then follows that if a distribution $F$ is given by an expression of the kind (20), such an expression is unique (on $\Sigma$).

If $T$ is a locally summable tensor, regularly discontinuous across $\Sigma$, the associated tensor-distribution $T^D$ has the following form,

$$T^D = T\chi^+_\Sigma + T\chi^-_\Sigma, \tag{23}$$

and one finds out (see [1, 4]) that the derivative of the associated tensor-distribution $T^D$ is different from the tensor-distribution associated with the derivative of $T$; in fact we have

$$\nabla(T^D) = (\nabla T)^D + \ell \otimes [T]\delta_\Sigma, \tag{24}$$

where $[T]$ denotes the jump of $T$ across $\Sigma$. Clearly, if $T$ is regular, i.e. $[T] = 0$, we have $\nabla(T^D) = (\nabla T)^D$.

Formula (24) can be iterated at any order, even if $\ell$ is only $C^1$ and $[T]$ is only well defined on $\Sigma$, by means of the method of regular prolongation and the consequent identification: $\nabla_A\ell[\nabla_B T] = [\nabla_A\ell\nabla_B T]$ for each pair of multi-indices $A = \alpha_1\cdots\alpha_n$ and $B = \beta_1\cdots\beta_m$ [11, 12]; it suffices to suppose that $f$ (and consequently $\ell$) is just regularly discontinuous, at any order of derivation, across $\Sigma$.

In the following, we will sometimes have to differentiate a product of the kind $\delta_\Sigma$ times a jump. This simply obeys the ordinary chain rule, thanks to our definition of covariant derivative on $\Sigma$ (section 2.2). To see this, consider the tensor-distribution $\nabla_\alpha(\delta_\Sigma[V_\beta])$ over a test tensor $U^{\alpha\beta}$. By definition of derivative of a distribution and of product between a function and a distribution, one has

$$\nabla_\alpha(\delta_\Sigma[V_\beta])(U^{\alpha\beta}) = -\delta_\Sigma([V_\beta]\nabla_\alpha U^{\alpha\beta}) \tag{25}$$

where, since $\delta_\Sigma$ has support on $\Sigma$, $\nabla_\alpha U^{\alpha\beta}$ is restricted to $\Sigma$, i.e. it is defined by means of the mean values of the Christoffel symbols. One then has

$$\begin{aligned}
[V_\beta]\nabla_\alpha U^{\alpha\beta} &= [V_\beta]\big(\partial_\alpha U^{\alpha\beta} + \overline{\Gamma}_{\alpha\nu}{}^{\beta}U^{\alpha\nu} + \overline{\Gamma}_{\alpha\nu}{}^{\alpha}U^{\nu\beta}\big)\\
&= [\partial_\alpha(V_\beta U^{\alpha\beta})] - U^{\alpha\beta}[\partial_\alpha V_\beta] + \overline{\Gamma}_{\alpha\nu}{}^{\alpha}[V_\beta]U^{\nu\beta} + \overline{\Gamma}_{\alpha\nu}{}^{\beta}[V_\beta]U^{\alpha\nu}\\
&= [\nabla_\alpha(V_\beta U^{\alpha\beta})] - \big[\Gamma_{\alpha\nu}{}^{\alpha}\big]\overline{V}_\beta U^{\nu\beta} - U^{\alpha\beta}\big([\nabla_\alpha V_\beta] + \big[\Gamma_{\alpha\beta}{}^{\sigma}\big]\overline{V}_\sigma\big)
\end{aligned} \tag{26}$$

which, by (3), is equal to

$$\nabla_\alpha[V_\beta U^{\alpha\beta}] - U^{\alpha\beta}\nabla_\alpha[V_\beta] \tag{27}$$

and since $\delta_\Sigma(\nabla_\alpha[V_\beta U^{\alpha\beta}])$ is equal to $-(\nabla_\alpha\delta_\Sigma)([V_\beta]U^{\alpha\beta})$, one concludes that

$$\nabla_\alpha(\delta_\Sigma[V_\beta]) = [V_\beta]\nabla_\alpha\delta_\Sigma + \nabla_\alpha[V_\beta]\delta_\Sigma \tag{28}$$

as wished.

### 2.5. Curvature tensor-distribution

Let us now introduce Lichnerowicz's *curvature tensor-distribution* $Q_{\alpha\beta\rho\sigma}$,

$$Q_{\alpha\beta\rho\sigma} \equiv (R_{\alpha\beta\rho\sigma})^D + H_{\alpha\beta\rho\sigma}\delta_\Sigma \tag{29}$$

where the singular component is defined as follows:

$$H_{\alpha\beta\rho\sigma} \equiv \ell_\beta[\Gamma_{\alpha\rho\sigma}] - \ell_\alpha[\Gamma_{\beta\rho\sigma}]. \tag{30}$$

The Christoffel symbols, under a generic regular coordinate change, transform according to the following non-tensorial law,

$$\Gamma_{\alpha\beta}{}^{\sigma} = \Gamma_{\alpha'\beta'}{}^{\sigma'}\frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial x^{\beta'}}{\partial x^\beta}\frac{\partial x^\sigma}{\partial x^{\sigma'}} + \frac{\partial^2 x^{\sigma'}}{\partial x^\alpha \partial x^\beta}\frac{\partial x^\sigma}{\partial x^{\sigma'}} \tag{31}$$

therefore, if we allow $C^2$ coordinate changes, we have

$$\left[\Gamma_{\alpha\beta}{}^{\sigma}\right] = \left[\Gamma_{\alpha'\beta'}{}^{\sigma'}\right]\frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial x^{\beta'}}{\partial x^\beta}\frac{\partial x^\sigma}{\partial x^{\sigma'}}, \tag{32}$$

i.e. the jump of the Christoffel symbols defines a tensor with support on $\Sigma$. Consequently, $H_{\alpha\beta\rho\sigma}$ is also a tensor with support on $\Sigma$, and therefore $Q_{\alpha\beta\rho\sigma}$ is a tensor-distribution. The role of the singular component $H_{\alpha\beta\rho\sigma}$ is to characterize gravitational shocks: in fact, such tensor vanishes if and only if the metric is $C^1$ across $\Sigma$; moreover, it follows from the weak formulation of the Einstein equations that its trace $H_{\beta\rho} = H_{\alpha\beta\rho}{}^{\alpha}$ necessarily vanishes unless the stress–energy content of the spacetime admits a singular component concentrated on $\Sigma$ [4, 5].

The curvature tensor-distribution (29) clearly satisfies the typical algebraic properties of a curvature tensor. As for the Ricci differential identities, it is possible to prove that they hold in the sense of distributions, i.e. for a regularly discontinuous vector we have (see [5] p 1511)

$$2\nabla_{[\beta}\nabla_{\alpha]}(V^\sigma)^D = \left(R_{\alpha\beta\rho}{}^{\sigma}V^\rho\right)^D + \delta_\Sigma H_{\alpha\beta\rho}{}^{\sigma}\overline{V}^\rho, \tag{33}$$

which, for a regular vector, reduces to

$$2\nabla_{[\beta}\nabla_{\alpha]}(V^\sigma)^D = Q_{\alpha\beta\rho}{}^{\sigma}V^\rho. \tag{34}$$

Moreover, it is possible to see that the Bianchi differential identities also hold, provided our definition (3) of covariant derivative on $\Sigma$ is used for the tensor $\left[\Gamma_{\alpha\rho}{}^{\sigma}\right]$ (see [5], theorem 6):

$$\nabla_{[\alpha}Q_{\beta\rho]\sigma\nu} = 0. \tag{35}$$

## 3. The wave operator in a curved spacetime

For a tensor $T$ of order $p$, the (generalized) Laplace operator is defined by (see, e.g., [14] and, with a different signature, [1] and [4] p 243)

$$(\Delta T)_{\alpha_1\cdots\alpha_p} = \nabla_\mu\nabla^\mu T_{\alpha_1\cdots\alpha_p} + \sum_{k=0}^{p} R_{\alpha_k\mu}T_{\alpha_1\cdots}{}^{\mu}{}_{\cdots\alpha_p} + \sum_{k=1,k\neq l}^{p} R_{\alpha_k\rho\alpha_l\sigma}T_{\alpha_1\cdots}{}^{\rho}{}_{\cdots}{}^{\sigma}{}_{\cdots\alpha_p} \tag{36}$$

where in the second term on the right-hand side $\mu$ is at the $k$th place, while in the third term $\rho$ and $\sigma$ are at the $k$th and $l$th place, respectively.

For example, for a scalar $u$, a vector $V$ and 2-tensor $T$, we have, respectively,

$$\begin{aligned}
\Delta u &= \nabla_\rho\nabla^\rho u \\
(\Delta V)_\alpha &= \nabla_\rho\nabla^\rho V_\alpha + R_\alpha{}^{\mu}V_\mu \\
(\Delta T)_{\alpha\beta} &= \nabla_\rho\nabla^\rho T_{\alpha\beta} + R_\alpha{}^{\mu}T_{\mu\beta} + R^{\mu}{}_{\beta}T_{\alpha\mu} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}T^{\mu\nu}.
\end{aligned} \tag{37}$$

Operator (36) reduces to the de Rham Laplacian [2] for antisymmetric tensors (see [1]). The following equivalent definitions for vectors and 2-tensors, in terms of commutators, are useful:

$$\begin{aligned}(\Delta V)_\alpha &= \nabla_\rho \nabla^\rho V_\alpha + (\nabla_\alpha \nabla_\rho - \nabla_\rho \nabla_\alpha) V^\rho \\ (\Delta T)_{\alpha\beta} &= \nabla_\rho \nabla^\rho T_{\alpha\beta} + (\nabla_\alpha \nabla_\sigma - \nabla_\sigma \nabla_\alpha) T^\sigma{}_\beta + (\nabla_\beta \nabla_\sigma - \nabla_\sigma \nabla_\beta) T_\alpha{}^\sigma .\end{aligned} \tag{38}$$

$\Delta$ is a linear hyperbolic and self-adjoint operator (see [1, 16]) and corresponds, but for an inessential factor $-1$, to the ordinary D'Alembert operator or wave operator $\Box = \partial_t^2 - \delta^{ik}\partial_i\partial_k$ for an ordinary function and in a flat spacetime.

Let us now consider the distribution $u^D$ associated with a regularly discontinuous function $u$. We can very easily extend the definition of the wave operator to such a field; it suffices to interpret the differential operators contained in (37) in the sense of distributions. We know how to handle such derivatives of $u^D$, thanks to (24); we have

$$\Delta u^D = \nabla_\rho \nabla^\rho u^D = \nabla_\rho ((\nabla^\rho u)^D + \ell^\rho [u] \delta_\Sigma), \tag{39}$$

and, by further application of (24) we obtain

$$\Delta u^D = (\Delta u)^D + (\ell^\rho [\nabla_\rho u] + \nabla_\rho ([u]\ell^\rho))\delta_\Sigma + (\ell \cdot \ell)[u]\delta'_\Sigma. \tag{40}$$

For a regularly discontinuous vector field $V$, from (24) we have

$$\nabla_\rho \nabla_\beta (V_\alpha)^D = (\nabla_\rho \nabla_\beta V_\alpha)^D + (\ell_\rho [\nabla_\beta V_\alpha] + \nabla_\rho ([V_\alpha]\ell_\beta))\delta_\Sigma + \ell_\rho \ell_\beta [V_\alpha]\delta'_\Sigma. \tag{41}$$

Thus from the definition in terms of commutators (38) we find

$$\begin{aligned}\Delta (V_\alpha)^D &= (\Delta V_\alpha)^D + (\ell^\rho [\nabla_\rho V_\alpha] + \nabla_\rho ([V_\alpha]\ell^\rho))\delta_\Sigma \\ &\quad + (\ell_\alpha ([\nabla_\rho V^\rho] - \nabla_\rho [V^\rho]) + \ell_\rho (\nabla_\alpha [V^\rho] - [\nabla_\alpha V^\rho]))\delta_\Sigma + (\ell \cdot \ell)[V_\alpha]\delta'_\Sigma.\end{aligned} \tag{42}$$

Now, from (3) we have

$$[\nabla_\rho V^\rho] - \nabla_\rho [V^\rho] = [\Gamma_{\rho\sigma}{}^\rho]\overline{V}^\sigma, \qquad \nabla_\alpha [V^\rho] - [\nabla_\alpha V^\rho] = -[\Gamma_{\alpha\sigma}{}^\rho]\overline{V}^\sigma, \tag{43}$$

and consequently, by (30),

$$\ell_\alpha ([\nabla_\rho V^\rho] - \nabla_\rho [V^\rho]) + \ell_\rho (\nabla_\alpha [V^\rho] - [\nabla_\alpha V^\rho]) = H_{\alpha\sigma}\overline{V}^\sigma, \tag{44}$$

where $H_{\alpha\sigma} \equiv H_{\rho\alpha\sigma}{}^\rho$. Thus equation (42) finally turns into

$$\Delta (V_\alpha)^D = (\Delta V_\alpha)^D + (\ell^\rho [\nabla_\rho V_\alpha] + \nabla_\rho ([V_\alpha]\ell^\rho) + H_{\alpha\sigma}\overline{V}^\sigma)\delta_\Sigma + (\ell \cdot \ell)[V_\alpha]\delta'_\Sigma. \tag{45}$$

Similarly, in the case of a regularly discontinuous 2-tensor field $T_{\alpha\beta}$, we have

$$\begin{aligned}\Delta (T_{\alpha\beta})^D &= (\Delta T_{\alpha\beta})^D + (\ell^\rho [\nabla_\rho T_{\alpha\beta}] + \nabla_\rho ([T_{\alpha\beta}\ell^\rho]))\delta_\Sigma + \big(\ell_\alpha \big([\nabla_\sigma T^\sigma{}_\beta] - \nabla_\sigma [T^\sigma{}_\beta]\big) \\ &\quad + \ell_\sigma \big(\nabla_\alpha [T^\sigma{}_\beta] - [\nabla_\alpha T^\sigma{}_\beta]\big)\big)\delta_\Sigma + \big(\ell_\sigma \big([\nabla_\beta T_\alpha{}^\sigma] - [\nabla_\beta T_\alpha{}^\sigma]\big) \\ &\quad + \ell_\beta \big([\nabla_\sigma T_\alpha{}^\sigma] - \nabla_\sigma [T_\alpha{}^\sigma]\big)\big)\delta_\Sigma + (\ell \cdot \ell)[T_{\alpha\beta}]\delta'_\Sigma,\end{aligned} \tag{46}$$

and again by (3) and (30) we, in the end, find

$$\begin{aligned}\Delta (T_{\alpha\beta})^D &= (\Delta T_{\alpha\beta})^D + (\ell^\rho [\nabla_\rho T_{\alpha\beta}] + \nabla_\rho ([T_{\alpha\beta}\ell^\rho]))\delta_\Sigma \\ &\quad + \big(H_{\alpha_\nu}\overline{T}^\nu{}_\beta + H_{\beta\nu}\overline{T}_\alpha{}^\nu + 2H_{\alpha\sigma\beta\nu}\overline{T}^{\sigma\nu}\big)\delta_\Sigma + (\ell \cdot \ell)[T_{\alpha\beta}]\delta'_\Sigma.\end{aligned} \tag{47}$$

We have examined the particularly significant cases of functions, vectors and 2-tensors; from the definition in terms of commutators (38) it follows that due to (33) it suffices to replace $R$ by $\underline{Q}$ in definition (37). Therefore for a generic $p$-tensor $T$ from (36) in the end we find

$$\begin{aligned}\Delta \big(T_{\alpha_1 \cdots \alpha_p}\big)^D &= \big(\Delta T_{\alpha_1 \cdots \alpha_p}\big)^D + \big(\ell^\rho \big[\nabla_\rho T_{\alpha_1 \cdots \alpha_p}\big] + \nabla_\rho \big(\big[T_{\alpha_1 \cdots \alpha_p}\ell^\rho\big]\big)\big)\delta_\Sigma \\ &\quad + \sum_{k=1, k\neq l}^{p} H_{\alpha_k \rho \alpha_l \sigma}\overline{T}_{\alpha_1 \ldots}{}^{\rho}{}_{\ldots}{}^{\sigma}{}_{\ldots \alpha_p}\delta_\Sigma + \sum_{k=0}^{p} H_{\alpha_k \mu}\overline{T}_{\alpha_1 \ldots}{}^{\mu}{}_{\ldots \alpha_p}\delta_\Sigma + (\ell \cdot \ell)\big[T_{\alpha_1 \cdots \alpha_p}\big]\delta'_\Sigma.\end{aligned} \tag{48}$$

The difference with the simple covariant generalization of the D'Alembert operator $\nabla^\alpha\nabla_\alpha$ is in the term involving the singular component of curvature distribution $H$; in fact we have

$$\nabla^\mu\nabla_\mu\big(T_{\alpha_1\cdots\alpha_p}\big)^D = \big(\Delta T_{\alpha_1\cdots\alpha_p}\big)^D + \big(\ell^\rho\big[\nabla_\rho T_{\alpha_1\cdots\alpha_p}\big] + \nabla_\rho\big(\big[T_{\alpha_1\cdots\alpha_p}\ell^\rho\big]\big)\big)\delta_\Sigma + (\ell\cdot\ell)\big[T_{\alpha_1\cdots\alpha_p}\big]\delta'_\Sigma. \tag{49}$$

Thus, different from $\nabla^\alpha\nabla_\alpha$, the generalized Laplace operator $\Delta$ in practice lets the singular structure of the spacetime be involved in the compatibility conditions for weak and singular solutions of the wave equation, as we will see in the following.

## 4. Weak solutions of the wave equation

Let $u$ be a regularly discontinuous function. We define $u$ as a weak solution of the wave equation if the equation holds for $u^D$, provided the differential operators are interpreted in the sense of distributions, i.e. if we have

$$\Delta u^D = 0. \tag{50}$$

By (40) we know that this equation is equivalent to

$$(\Delta u)^D + (\ell^\rho[\nabla_\rho u] + \nabla_\rho([u]\ell^\rho))\delta_\Sigma + (\ell\cdot\ell)[u]\delta'_\Sigma = 0. \tag{51}$$

However, from (51) it follows $(\Delta u)^D = 0$ (to see this it suffices to consider separately the set of test functions with support $K \subset \Omega\backslash\Sigma$); consequently, from the properties of independence of $\delta_\Sigma$ and $\delta'_\Sigma$ (see section 2.4) the weak equation (50) is actually equivalent to the following set:

$$(\Delta u)^D = 0, \qquad \ell^\rho[\nabla_\rho u] + \nabla_\rho([u]\ell^\rho) = 0, \qquad (\ell\cdot\ell)[u] = 0. \tag{52}$$

Equation $(\Delta u)^D = 0$ is simply equivalent to the ordinary wave equation to hold on each side of $\Omega$, separately (to see this, it suffices to consider the class of test functions with support $K \subset \Omega^-$ and then those with support $K \subset \Omega^+$); the further two conditions are actually compatibility conditions which must hold on $\Sigma$. Thus a regularly discontinuous function $u$ is a weak solution of the wave equation if and only if it is a solution on each side of the discontinuity hypersurface and, moreover, we have

$$\ell^\rho[\nabla_\rho u] + \nabla_\rho([u]\ell^\rho) = 0 \tag{53}$$

and $(\ell\cdot\ell)[u] = 0$. In a similar way, if we define a regularly discontinuous vector $V$ as a weak solution of the wave equation by $\Delta V^D = 0$, then from (45) we find that $V$ is a weak solution if and only if it is a solution on each side of the discontinuity hypersurface and the following compatibility condition holds on $\Sigma$,

$$\ell^\rho[\nabla_\rho V_\alpha] + \nabla_\rho([V_\alpha]\ell^\rho) + H_{\alpha\sigma}\overline{V}^\sigma = 0, \tag{54}$$

with, moreover, $(\ell\cdot\ell)[V_\alpha] = 0$. Similarly, we see from (47) that a regularly discontinuous 2-tensor $T$ is a weak solution if and only if it is a solution on each side of the discontinuity hypersurface, and the following compatibility condition holds on $\Sigma$,

$$\ell^\rho[\nabla_\rho T_{\alpha\beta}] + \nabla_\rho([T_{\alpha\beta}\ell^\rho]) + \big(H_{\alpha\nu}\overline{T}^\nu{}_\beta + H_{\beta\nu}\overline{T}_\alpha{}^\nu + 2H_{\alpha\sigma\beta\nu}\overline{T}^{\sigma\nu}\big) = 0 \tag{55}$$

and, moreover, $(\ell\cdot\ell)[T_{\alpha\beta}] = 0$. We note that in any case we must have $(\ell\cdot\ell) = 0$ for the discontinuity to be present (otherwise the jump is null and the discontinuity hypersurface is such only for the derivatives of the field). This means that, as expected, discontinuity hypersurfaces for the wave equation must be characteristic, i.e. lightlike (just like the characteristics of the Einstein equations).

For vectors and tensors we, moreover, note that compatibility conditions also involve the mean value of the field on the discontinuity hypersurface, and not only its jump, like in the case of ordinary functions. This seems to be a novelty due to the generality of the present approach. This kind of contribution disappears if the metric is $C^1$.

We have examined in some detail the particularly significant cases of functions, vectors and 2-tensors; turning to the general case of a regularly discontinuous $p$-tensor $T$, $p \geqslant 0$, again we obviously find the characteristic condition $(\ell \cdot \ell)[T_{\alpha_1\cdots\alpha_p}] = 0$, plus the following compatibility condition:

$$\ell^\rho\left[\nabla_\rho T_{\alpha_1\cdots\alpha_p}\right] + \nabla_\rho\left(\left[T_{\alpha_1\cdots\alpha_p}\ell^\rho\right]\right) + \sum_{k=0}^{p} H_{\alpha_k\mu}\overline{T}_{\alpha_1\cdots}{}^{\mu}{}_{\cdots\alpha_p} + \sum_{k=1,k\neq l}^{p} H_{\alpha_k\rho\alpha_l\sigma}\overline{T}_{\alpha_1\cdots}{}^{\rho}{}_{\cdots}{}^{\sigma}{}_{\cdots\alpha_p} = 0. \tag{56}$$

We thus have proved the following theorem.

**Theorem 1.** *A regularly discontinuous tensor field $T$ is a non-trivial weak solution of the wave equation $\Delta T^D = 0$ if and only if $\Sigma$ is characteristic, $T$ is a solution of the wave equation in the ordinary sense on each side of $\Sigma$ separately, and compatibility condition (56) holds.*

## 5. Singular solutions of the wave equation with support on a hypersurface

Let $u$ be a singular distribution; we say that $u$ is a singular solution of the wave equation if it satisfies such equation in the sense of distributions. Let us consider singular distributions with support on a hypersurface $\Sigma$, with the following expression:

$$u = \gamma\delta_\Sigma. \tag{57}$$

Such kind of distribution defines a field concentrated on $\Sigma$, where the scalar field $\gamma \in C^h(\Omega)$, $h \geqslant 2$, is the singular (here unique) component of the field.

The choice to consider $\gamma$ as defined (and regular) in a neighbourhood of $\Sigma$ is clearly helpful for handling with differential operators. However, since the value of $u$ on a generic test function is influenced by the restriction $\gamma|_\Sigma$ of $\gamma$ on $\Sigma$ only, one may legitimately demand that a singular field with support on $\Sigma$ should have its component $\gamma$ defined on $\Sigma$ only. In this case, our choice can be viewed as an operational choice if we assume that the values of $\gamma$, and also those of its derivatives up to order 2, are well defined on $\Sigma$ only, so that, however, it is possible to consider arbitrary regular prolongations $\gamma \in C^h(\Omega)$ (provided their values coincide on $\Sigma$, together with those of their derivatives). The differential equations we obtain in the following for $\gamma$ then have two possible interpretations: (a) only their restriction on $\Sigma$ (where differential operators on $\Sigma$ follows the definitions of section 2.2) properly holds; (b) they actually select the prolongation, or a class of admissible prolongations.

We can in any case handle differential operators applied to concentrated fields, thanks to (17) and to the following similar formula for the derivatives of $\delta'_\Sigma$,

$$\nabla\delta'_\Sigma = \ell\delta''_\Sigma, \tag{58}$$

which involves a further singular distribution with support on $\Sigma$, denoted by $\delta''_\Sigma$. Let us see that (58) holds; in a generic chart, by derivation of (17) we have

$$\nabla_\alpha\nabla_\beta\delta_\Sigma = \nabla_\alpha\ell_\beta\delta'_\Sigma + \ell_\alpha\nabla_\beta\delta'_\Sigma. \tag{59}$$

The left-hand side of the above formula is symmetric with respect to the pair of indices $\alpha$ and $\beta$ since $\delta_\Sigma$ is a scalar-distribution. On the right-hand side, $\nabla\ell$ is symmetric, since $\ell$ is a gradient and $\varphi$ is $C^2$. Thus we have

$$\ell_\alpha\nabla_\beta\delta'_\Sigma = \ell_\beta\nabla_\alpha\delta'_\Sigma. \tag{60}$$

Since $\Sigma$ is regular and $\ell$ is non-null, we can fix an index, say $\bar{\alpha}$, such that $\ell_{\bar{\alpha}} \neq 0$. We then have

$$\nabla_\beta \delta'_\Sigma = \ell_\beta {\ell_{\bar{\alpha}}}^{-1} \nabla_{\bar{\alpha}} \delta'_\Sigma. \tag{61}$$

Thus we have that the expression ${\ell_\alpha}^{-1} \nabla_\alpha \delta'_\Sigma$ is invariant with respect to the choice of $\alpha$ such that $\ell_\alpha \neq 0$, and we can use it as the definition of the distribution $\delta''_\Sigma$; therefore (61) now reduces to (58), as wished.

Moreover, it is easy to see that $\delta''_\Sigma$ is also independent of $\delta_\Sigma$ and $\delta'_\Sigma$ in the sense of section 2.4.

We can now apply (17) and (58) and obtain

$$\Delta(\gamma\delta_\Sigma) = \Delta\gamma\delta_\Sigma + (\ell^\rho \nabla_\rho \gamma + \nabla_\rho(\gamma \ell^\rho))\delta'_\Sigma + (\ell \cdot \ell)\gamma \delta''_\Sigma. \tag{62}$$

Therefore the wave equation for our concentrated field

$$\Delta(\gamma\delta_\Sigma) = 0 \tag{63}$$

is equivalent to the following set:

$$\Delta\gamma = 0, \qquad \ell^\rho \nabla_\rho \gamma + \nabla_\rho(\gamma\ell^\rho) = 0, \qquad (\ell \cdot \ell)\gamma = 0. \tag{64}$$

We thus have proved that $\gamma\delta_\Sigma$ is a singular solution of the wave equation if and only if $\gamma$ is a solution of such equation (in a neighbourhood of $\Sigma$) and, moreover, the following compatibility condition holds:

$$\ell^\rho \nabla_\rho \gamma + \nabla_\rho(\gamma\ell^\rho) = 0. \tag{65}$$

Moreover, since $(\ell \cdot \ell)\gamma = 0$, the singular hypersurface $\Sigma$ must be characteristic, otherwise $\gamma = 0$ and our solution vanishes.

Similar considerations hold for the case of a singular concentrated vector solution $V_\alpha \delta_\Sigma$; we have the following compatibility condition,

$$\ell^\rho \nabla_\rho V_\alpha + \nabla_\rho(V_\alpha \ell^\rho) = 0, \tag{66}$$

and $(\ell \cdot \ell)V_\alpha = 0$. For the case of a singular concentrated 2-tensor solution $T_{\alpha\beta}\delta_\Sigma$ we again have

$$\ell^\rho \nabla_\rho T_{\alpha\beta} + \nabla_\rho(T_{\alpha\beta}\ell^\rho) = 0 \tag{67}$$

and $(\ell \cdot \ell)T_{\alpha\beta} = 0$. Finally, in the general case, for a generic concentrated tensor of order $p \geqslant 0$ again we find the characteristic condition $(\ell \cdot \ell)T_{\alpha_1 \cdots \alpha_p} = 0$, plus the following compatibility condition:

$$\ell^\rho \nabla_\rho T_{\alpha_1 \cdots \alpha_p} + \nabla_\rho\left(T_{\alpha_1 \cdots \alpha_p}\ell^\rho\right) = 0. \tag{68}$$

In particular, in all cases we find that $\Sigma$ must be characteristic in order for the singular solution to be non-trivial, and that, different from weak solutions, for singular solutions the curvature tensor-distribution is not involved in the compatibility conditions. We thus have proved the following theorem.

**Theorem 2.** *A concentrated tensor field $T\delta_\Sigma$, where $T$ is a p-tensor field of class $C^h(\Omega)$, $h \geqslant 2$, is a non-trivial singular solution of the wave equation $\Delta(T\delta_\Sigma) = 0$ if and only if $\Sigma$ is characteristic, $T$ is a solution of the wave equation in the ordinary sense in a neighbourhood of $\Sigma$, and compatibility condition (68) holds.*

## 6. The wave equation with source: an application to electromagnetism

### *6.1. Maxwell equations and the wave equation*

Let us consider the relativistic Maxwell differential system,

$$\nabla_{[\rho}F_{\alpha\beta]} = 0, \qquad \nabla_\alpha F^{\alpha\beta} = J^\beta, \tag{69}$$

where $F$ is the electromagnetic field tensor and $J$ is the charge-current vector, which is conservative

$$\nabla_\alpha J^\alpha = 0. \tag{70}$$

The unit volume 4-form (Ricci antisymmetric tensor) $\eta = \sqrt{|g|}\epsilon$, where $\epsilon$ is the Levi-Civita indicator, permits us to define the antisymmetric dual of a given antisymmetric 2-tensor in a standard way. The dual of the electromagnetic field is

$$(*F)^{\alpha\beta} = (1/2)\eta^{\alpha\beta}{}_{\mu\nu}F^{\mu\nu}. \tag{71}$$

The dual operator is involutive. Some useful properties of $\eta$ are

$$\begin{aligned}&(1/2)\eta_{\alpha\rho\lambda\mu}\eta_\beta{}^\rho{}_{\sigma\nu} = g_{\alpha\beta}g_{\mu[\sigma}g_{\nu]\lambda} + g_{\lambda\beta}g_{\alpha[\sigma}g_{\nu]\mu} + g_{\mu\beta}g_{\lambda[\sigma}g_{\nu]\alpha}\\ &(1/2)\eta_{\rho\sigma\lambda\mu}\eta^{\rho\sigma}{}_{\alpha\beta} = g_{\mu\alpha}g_{\lambda\beta} - g_{\lambda\alpha}g_{\mu\beta}.\end{aligned} \tag{72}$$

Consider $\nabla_\alpha(*F)^{\alpha\beta}$; from (72), we have the following identity,

$$\begin{aligned}\eta_\beta{}^{\rho\sigma\gamma}\nabla_\alpha(*F)^{\alpha\beta} &= (1/2)\eta^{\rho\beta\gamma\sigma}\eta^\alpha{}_\beta{}^{\mu\nu}\nabla_\alpha F_{\mu\nu}\\ &= \nabla^\rho F^{\sigma\gamma} + \nabla^\sigma F^{\gamma\rho} + \nabla^\gamma F^{\rho\sigma}\end{aligned} \tag{73}$$

and similarly for $\eta_\beta{}^{\rho\sigma\gamma}\nabla_\alpha F^{\alpha\beta}$, replacing $F$ with $(*F)$. Consequently, an equivalent formulation of system (69) is as follows:

$$\nabla_\alpha(*F)^\alpha{}_\beta = 0, \qquad \nabla_\alpha F^\alpha{}_\beta = J^\beta. \tag{74}$$

Now from identity (73) we have

$$\eta_\beta{}^{\rho\sigma\gamma}\nabla_\rho\nabla_\alpha(*F)^{\alpha\beta} = \nabla_\rho\nabla^\rho F^{\sigma\gamma} + \nabla_\rho\nabla^\sigma F^{\gamma\rho} + \nabla_\rho\nabla^\gamma F^{\rho\sigma}.$$

Using the Ricci formula for the inversion of iterated covariant derivatives and the antisymmetry of $F$, we equivalently have

$$\eta_\beta{}^{\rho\sigma\gamma}\nabla_\rho\nabla_\alpha(*F)^{\alpha\beta} = (\Delta F)^{\sigma\gamma} + \nabla^\sigma\nabla_\rho F^{\gamma\rho} + \nabla^\gamma\nabla_\rho F^{\rho\sigma}$$

and similarly for $\eta_\beta{}^{\rho\sigma\gamma}\nabla_\rho\nabla_\alpha F^{\alpha\beta}$, replacing $F$ with $(*F)$. Now if (74) holds we have

$$(\Delta F)^{\sigma\gamma} = \nabla^\sigma J^\gamma - \nabla^\gamma J^\sigma \tag{75}$$

or, equivalently, in terms of $(*F)$,

$$[\Delta(*F)]^{\lambda\delta} = \eta^{\lambda\delta\sigma\gamma}\nabla_\sigma J_\gamma. \tag{76}$$

In case (74) holds, the equivalent relations (75) and (76) are identities. However, each of them as a differential equation is also equivalent, in a sense, to system (69). In fact, if we denote $D^\beta = \nabla_\alpha(*F)^{\alpha\beta}$ and $E^\beta = \nabla_\alpha F^{\alpha\beta} - J^\beta$, they satisfy the following conditions, as we will see in what follows:

$$(\Delta D)_\mu = 0, \qquad (\Delta E)_\mu = 0. \tag{77}$$

Now, since $\Delta$ is a linear hyperbolic and self-adjoint operator, if we work in the framework of tensor-distributions with a compact support (or, more generally, in that of tensor-distributions with a past compact (future compact) support, see [1, 15]) we have that (77) implies $E = D = 0$ (see [1, 14–16]). If we instead work in the framework of regular $C^\infty$ ordinary functions, in the domain of a fixed local chart, from (77) we again have that if $D$ and $E$ are null at a given initial manifold (i.e., if the Maxwell equations (69) hold initially) then $D = E = 0$ always [15]. In any case this means that solutions of (75) (or, equivalently (76)) are solutions of (69). This is a (heuristic at least) reason for adopting (75) as a possible alternative to Maxwell equations. And, in fact, in the following we will study weak and singular solutions of (75) according to the theory stated above. Let us now see that (77) hold; we have

$$\eta^{\beta\rho\sigma\gamma} D_\beta = \nabla^\rho F^{\sigma\gamma} + \nabla^\gamma F^{\sigma\rho} + \nabla^\sigma F^{\gamma\rho}$$

and from (75) $\eta^{\beta\rho\sigma\gamma}\nabla_\rho D_\beta = 0$. By saturation with $\eta_{\sigma\gamma}{}^{\lambda\mu}$ we thus have $\nabla^{[\lambda} D^{\mu]} = 0$. Consequently, $D$ is a solution of the equation $(\Delta D)^\mu - \nabla^\mu\nabla_\lambda D^\lambda = 0$, and since $D$ is divergence-free, it is also a solution of equation $(77)_1$. Similarly, from the equivalent equation (76) we can show that we necessarily have $(77)_2$.

### 6.2. Discontinuous electromagnetic field without source

Let us consider (75). If $J = 0$ the electromagnetic field is radiative, i.e. it satisfies the homogeneous wave equation. In such a case, we can consider weak electromagnetic solutions characterized by a regularly discontinuous $F$ such that

$$\Delta(F_{\alpha\beta})^D = 0, \tag{78}$$

which from theorem 1 must have a characteristic discontinuity hypersurface (wavefront) $\Sigma$, and a compatibility condition of the kind (55) holding on $\Sigma$,

$$\ell^\rho[\nabla_\rho F_{\alpha\beta}] + \nabla_\rho([F_{\alpha\beta}\ell^\rho]) + H_{\alpha_\nu}\overline{F}^\nu{}_\beta + H_{\beta\nu}\overline{F}_\alpha{}^\nu + 2H_{\alpha\sigma\beta\nu}\overline{F}^{\sigma\nu} = 0. \tag{79}$$

Such a condition explicitly involves the singular component $H$ of the curvature tensor-distribution, therefore (79) governs in a sense of the general interaction of electromagnetic and gravitational shocks; however, if a gravitational shock wave is present with $\Sigma$ as wavefront (and no stress–energy is concentrated on $\Sigma$), then $H_{\alpha\beta\rho\sigma} \neq 0$ and $H_{\alpha\beta} = 0$, thus we have

$$\ell^\rho[\nabla_\rho F_{\alpha\beta}] + \nabla_\rho([F_{\alpha\beta}\ell^\rho]) + 2H_{\alpha\sigma\beta\nu}\overline{F}^{\sigma\nu} = 0. \tag{80}$$

In the following, we will consider a weak electromagnetic solution with a non-null (generalized) charge-current source. This can be treated quite easily with our method, with the obvious appearance of additional terms. For a regularly discontinuous charge-current vector $J$ we have, from (24),

$$\nabla_\alpha(J_\beta)^D - \nabla_\beta(J_\alpha)^D = (\nabla_\alpha J_\beta - \nabla_\beta J_\alpha)^D + (\ell_\alpha[J_\beta] - \ell_\beta[J]_\alpha)\delta_\Sigma, \tag{81}$$

and for a concentrated charge-current component $\hat{J}\delta_\Sigma$, from (17), we have

$$\nabla_\alpha(\hat{J}_\beta\delta_\Sigma) - \nabla_\beta(\hat{J}_\alpha\delta_\Sigma) = (\nabla_\alpha\hat{J}_\beta - \nabla_\beta\hat{J}_\alpha)\delta_\Sigma + (\ell_\alpha\hat{J}_\beta - \ell_\beta\hat{J}_\alpha)\delta'_\Sigma. \tag{82}$$

The condition of conservation (70), imposed on $J^D$ and $\hat{J}$, implies that $J$ is conservative at each side of $\Sigma$ separately, that $\hat{J}$ is conservative in a neighbourhood of $\Sigma$, and that the following conditions hold at $\Sigma$:

$$[J_\alpha]\ell^\alpha = 0, \qquad \hat{J}_\alpha\ell^\alpha = 0. \tag{83}$$

On the other hand, for a regularly discontinuous electromagnetic field $F$ we have, from (47),

$$\begin{aligned}\Delta(F_{\alpha\beta})^D &= (\Delta F_{\alpha\beta})^D + (\ell^\rho[\nabla_\rho F_{\alpha\beta}] + \nabla_\rho([F_{\alpha\beta}\ell^\rho]))\delta_\Sigma \\ &\quad + \left(H_{\alpha_\nu}\overline{F}^\nu{}_\beta + H_{\beta\nu}\overline{F}_\alpha{}^\nu + 2H_{\alpha\sigma\beta\nu}\overline{F}^{\sigma\nu}\right)\delta_\Sigma + (\ell\cdot\ell)[F_{\alpha\beta}]\delta'_\Sigma.\end{aligned} \tag{84}$$

Similarly, for a singular electromagnetic field $\hat{F}\delta_\Sigma$ we have

$$\Delta(\hat{F}_{\alpha\beta}\delta_\Sigma) = (\Delta\hat{F}_{\alpha\beta})\delta_\Sigma + (\ell^\rho\nabla_\rho\hat{F}_{\alpha\beta} + \nabla_\rho(\hat{F}_{\alpha\beta}\ell^\rho))\delta'_\Sigma + (\ell\cdot\ell)\hat{F}_{\alpha\beta}\delta''_\Sigma. \tag{85}$$

### 6.3. Discontinuous electromagnetic field with discontinuous source

In the presence of regularly discontinuous charge-current, we define $F$ as a weak solution of equation (75) if we have

$$\Delta(F_{\alpha\beta})^D = \nabla_\alpha(J_\beta)^D - \nabla_\beta(J_\alpha)^D. \tag{86}$$

Therefore, from comparison of (81) and (84) we have that in this case $F$ is a solution of (75) on each side of $\Sigma$ separately, that the discontinuity hypersurface $\Sigma$ is characteristic, and that, moreover, the following compatibility condition holds on $\Sigma$:

$$\ell^\rho[\nabla_\rho F_{\alpha\beta}] + \nabla_\rho[F_{\alpha\beta}\ell^\rho] + 2H_{\nu[\alpha}\overline{F}^{\nu}{}_{\beta]} + 2H_{\alpha\sigma\beta\nu}\overline{F}^{\sigma\nu} = \ell_\alpha[J_\beta] - \ell_\beta[J_\alpha]. \tag{87}$$

Therefore, if a gravitational shock wave is present, we have

$$\ell^\rho[\nabla_\rho F_{\alpha\beta}] + \nabla_\rho[F_{\alpha\beta}\ell^\rho] + 2H_{\alpha\sigma\beta\nu}\overline{F}^{\sigma\nu} = \ell_\alpha[J_\beta] - \ell_\beta[J_\alpha], \tag{88}$$

which corresponds to Lichnerowicz's propagation formula (IV.8.6) in [4], where it was obtained directly, starting from the Maxwell equations and using harmonic techniques; this correspondence is a test of physical significance that our method has successfully got through.

### 6.4. Discontinuous electromagnetic field with singular charge-current

If instead the charge-current is concentrated on the hypersurface $\Sigma$, we cannot discard the further interaction components which appear in (87), since in this case we have $H_{\alpha\beta} \neq 0$. A regularly discontinuous weak solution of equation (75) is in this case defined by

$$\Delta(F_{\alpha\beta})^D = \nabla_\alpha(\hat{J}_\beta\delta_\Sigma) - \nabla_\beta(\hat{J}_\alpha\delta_\Sigma), \tag{89}$$

and from comparison of (82) and (84) we find that $F$ is a solution of the wave equation $\Delta F = 0$ on each side of $\Sigma$ separately, and that on $\Sigma$ we have

$$\begin{aligned} &\ell^\rho[\nabla_\rho F_{\alpha\beta}] + \nabla_\rho[F_{\alpha\beta}\ell^\rho] + 2H_{\nu[\alpha}\overline{F}^{\nu}{}_{\beta]} + 2H_{\alpha\sigma\beta\nu}\overline{F}^{\sigma\nu} = \nabla_\alpha\hat{J}_\beta - \nabla_\beta\hat{J}_\alpha, \\ &(\ell\cdot\ell)[F_{\alpha\beta}] = \ell_\alpha\hat{J}_\beta - \ell_\beta\hat{J}_\alpha. \end{aligned} \tag{90}$$

In particular, $\Sigma$ is not necessarily characteristic in this case; this means that the singular electromagnetic source shell can evolve in time according to the causality condition.

### 6.5. Singular electromagnetic field with singular charge-current

Finally, let us consider the case of a singular electromagnetic field $F\delta_\Sigma$ corresponding to a singular charge-current, $\hat{J}\delta_\Sigma$:

$$\Delta(\hat{F}_{\alpha\beta}\delta_\Sigma) = \nabla_\alpha(\hat{J}_\beta\delta_\Sigma) - \nabla_\beta(\hat{J}_\alpha\delta_\Sigma). \tag{91}$$

From comparison of (82) and (85) we have

$$\Delta\hat{F}_{\alpha\beta} = \nabla_\alpha\hat{J}_\beta - \nabla_\beta\hat{J}_\alpha, \qquad \ell^\rho\nabla_\rho\hat{F}_{\alpha\beta} + \nabla_\rho(\hat{F}_{\alpha\beta}\ell^\rho) = \ell_\alpha\hat{J}_\beta - \ell_\beta\hat{J}_\alpha, \qquad (\ell\cdot\ell)\hat{F}_{\alpha\beta} = 0. \tag{92}$$

In particular, again $\Sigma$ must be characteristic in order for $\hat{F}$ to be non-null. More general situations, involving electromagnetic fields and charge-currents with both a regularly discontinuous component and a singular component, and even with the presence of more singular components like those along $\delta''_\Sigma$, can be studied along the same lines.

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